ENERGIZE YOUR MATH CLASS WITH MATHEMATICAL MODELS

Robert E. Kowalczyk and Adam O. Hausknecht
University of Massachusetts Dartmouth
Mathematics Department, 285 Old Westport Road, N. Dartmouth, MA 02747-2300
rkowalczyk@umassd.edu and ahausknecht@umassd.edu

Whether you're teaching a general education mathematics course or a calculus course, you need to change the pace of the course every so often to maintain your students' interest. Introducing a creative mathematical model is a great way of doing this. For example, one week you can bring to class a hot cup of coffee and work with the students to model the temperature of the coffee over time. During another week you can take your class outdoors and measure the height of the campus campanile (bell tower) or you can model how long it takes *n* students to cheer "Go Patriots" (Super Bowl XXXVI football champions). In this paper, we will demonstrate how to use digital cameras, video, and computer software to develop and implement mathematical models that grab your students' interest and keep them focused on the usefulness of mathematics. With these experimental math models, students become involved in the class by taking part in some physical activity. Rather than sitting at their desks the entire class period being passive scribes, they physically take part in the class, see the power and usefulness of mathematical models, and, hopefully, are motivated to learn mathematics and even possibly encouraged to take another mathematics course. Some example models are given below.

Modeling the Depth of Water in a Vase

In many of the standard first-year calculus books (e. g., Stewart, Larsen, Hughs-Hallett, Tan), there is a common exercise that asks students to draw a rough sketch of the depth of water in a vase that is being filled at a constant rate. We have given this exercise to our students as a class activity and also as a problem on a test. Our students struggle to visualize the rate of change of the depth of the water and most of the time they draw an incorrect graph. To help our students better visualize this situation, we decided to video a clear glass vase being filled with water. By observing this video and its subsequent analysis, students will develop an improved intuition and understanding of rates of change. We also wanted our students to check how well their hand-drawn graphs of the depth of water over time models the actual experiment. To do this, we used the software package "Graphic Converter[1]" to convert the video into a sequence of individual frames equally spaced in time. This sequence of digital images was then imported into the TEMATH[2] software package and the first image was positioned over a pair of coordinate axes (see Figure 1 — red food coloring was added to the water to make the water level more visible). Using TEMATH's Point tool, students can mark the depth of the water by clicking a coordinate point at the top edge of the water level, thus, measuring the depth of the water. Next, they can move frame by frame through all the digital images and mark the

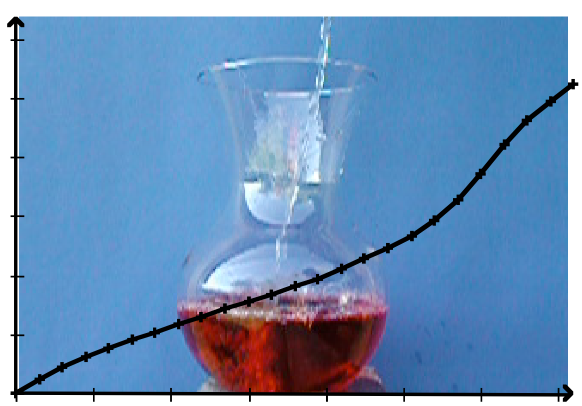

Figure 1 Modeling the Depth of Water

depth of the water as a function of time. The measured depths are then placed into a table and plotted as a function of time. Students can observe the plotted data points and easily determine the concavity of the depth curve and the position of the points of inflection.

Generating a Cycloid

For this model, we took a cylindrical can, marked a point on the circumference of its bottom, rolled it along a flat surface, and made a video of the rolling can. Theory tells us that the point on the can's rim traces a path called a *cycloid*. The parametric equations that model the cycloid are

$$x(\theta) = r(\theta - \sin(\theta)),\ y(\theta) = r(1 - \cos(\theta))$$

where r is the radius of the circular bottom of the can and θ is the angle of rotation of the rolling can. Graphic Converter was used to convert the video into a sequence of individual frames and the frames were imported into TEMATH. The first image was positioned over a pair of coordinate axes. Using TEMATH's Line tool, students can measure the radius r of the can from one of the frames of the video. The standard parametric equations for the cycloid assume that $x(0) = y(0) = 0$, that is, the cycloid "begins" at the origin. However, the first frame of the video doesn't necessarily show the marked point at the origin. Thus, the frame must be moved so that the point is aligned with the origin, or, as we prefer, TEMATH's Line tool can be used to measure the horizontal shift between the origin and the point on the can. If the horizontal shift is measured, then the parametric equation for x becomes $x(\theta) = r(\theta - \sin(\theta)) + h$, where h is the horizontal shift. Once the radius r and the horizontal shift h are measured, the parametric equations are entered into TEMATH and plotted on top of the image of the rolling can. To check how well the theory models the experiment, students can hold down the right arrow key on the keyboard and display the images in a rapid sequence simulating a video. If the measured parameters are accurate, students will see the point on the rolling can tracing the plotted parametric cycloid. See Figure 2 for a single frame of the video and the plotted cycloid.

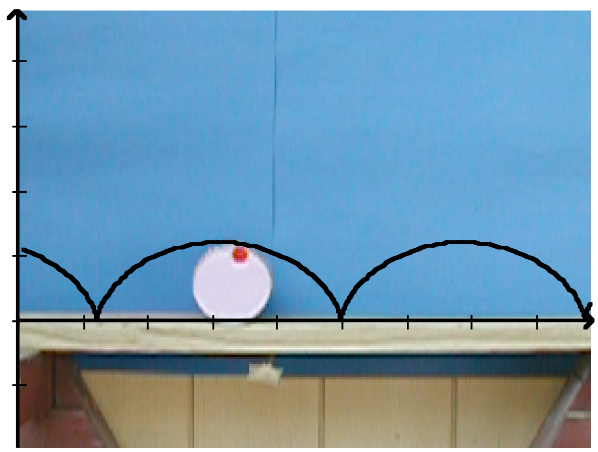

Figure 2 Modeling a Cycloid

Periodic Motion

This experiment can be used in a general education mathematics class, a calculus class, or a differential equations class when studying periodic functions or harmonic motion. For this experiment, we set up a mass-spring system, put it into motion, and recorded a video of the periodic motion of the mass. After importing the frames of the video into TEMATH, we used TEMATH's Point tool to mark the position of the mass in each frame. The x-coordinate of the positions were placed into a table and plotted as a function of time. The pattern of the plotted points was a sinusoid of the form

$$f(t) = a\sin\left(\frac{2\pi}{T}(t - b)\right) + c$$

where a is the amplitude, T is the period, b is the horizontal shift, and c is the vertical shift. Using the plotted points and TEMATH's Line tool, students estimated the four parameters of the sinusoid. The sinusoid was then plotted on top of the points to determine how well it models the periodic motion of the mass-spring system. Figure 3 shows the great fit of the model to the measured data.

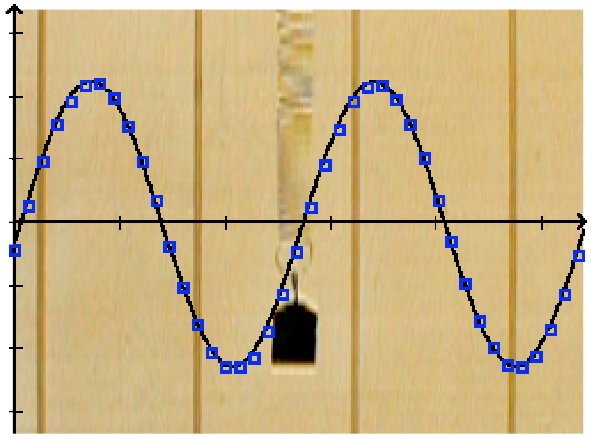

Figure 3 Modeling Periodic Motion

Damped Harmonic Motion

As an extension to the Periodic Motion experiment, we placed the mass of the mass-spring system into a clear container filled with water. The water causes the system to become damped. We then set this damped system into motion, recorded a video, and imported its frames into TEMATH. The measured positions of the mass were plotted resulting in a pattern of the exponentially damped sinusoid

$$f(t) = ae^{-kt}\sin\left(\frac{2\pi}{T}(t - b)\right) + c$$

where a is the amplitude, k is the damping coefficient, T is the period, b is the horizontal shift, and c is the vertical shift. In this experiment, students must model both an exponential function and a sinusoid. An example model and data are shown in Figure 4. This is an excellent model to use with students when studying periodic functions or exponential functions.

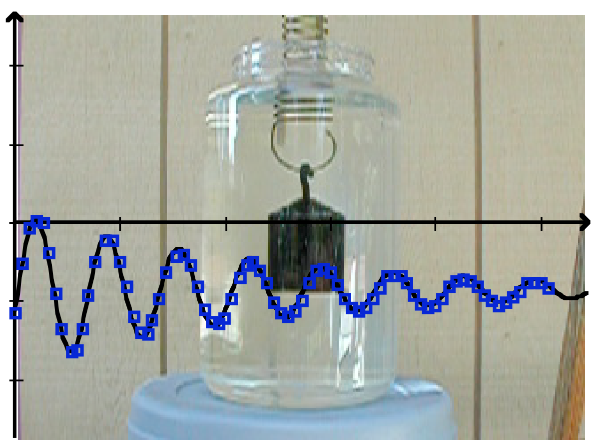

Figure 4 Damped Harmonic Motion

Logistic Growth of a Stain

The logistic growth function is now one of the standard functions presented in most first-year mathematics books. Most of the logistic data given in these books for examples and exercises are various types of population growth data obtained from referenced sources. We wanted a hands-on experiment for our students so that they could really visualize logistic growth — the mathematical expression for logistic growth is rather complicated and not very revealing to a first-year student. After much brainstorming, we decided on an experiment that models the spreading of a stain. Using an eyedropper, some colored water, and a paper towel, we made a video of the colored water being dropped onto the paper towel and the resulting spread of the colored stain on the towel. The frames of the video were imported into TEMATH and the Circle function was used to measure the area of the stain in each frame (see Figure 5). The measured areas were plotted as a function of time resulting in a curve resembling logistic growth. The mathematical model for the logistic growth of the area is $A(t) = \dfrac{L}{1 + ae^{-bt}}$, where L is the limiting value, a and b are parameters determining the rate of growth, and t is time. L is fairly easy to estimate. However, a and b are much more difficult to estimate. After some trial and error, students will become familiar with how different values of a and b change the shape of the logistic curve. An example model and measured data are shown in Figure 6.

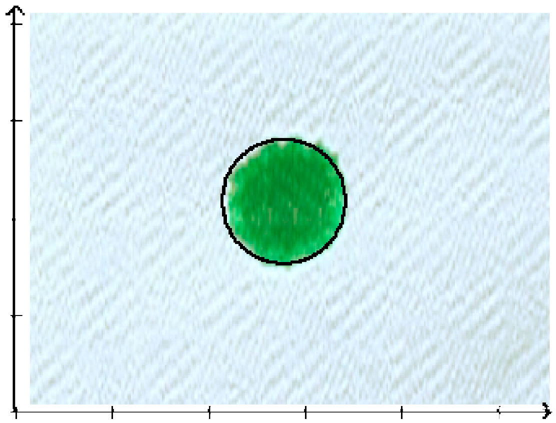

Figure 5 Using the Circle Function

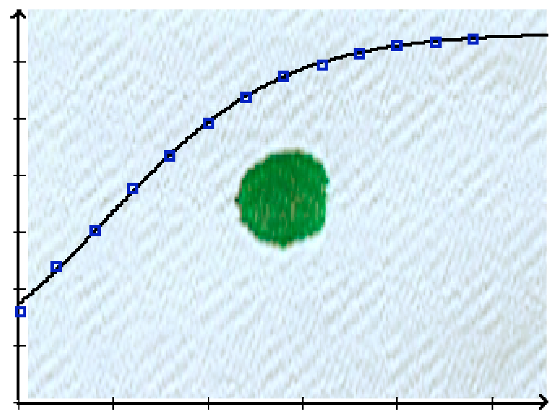

Figure 6 Logistic Growth of a Stain

Modeling Free Fall Flight and Terminal Velocity

For this exploration, we selected a coffee filter for our free fall flight. A coffee filter is light enough to reach a terminal velocity quickly and a number of filters can be stacked together to change the mass of the falling object. Using a ladder, we dropped the coffee filters from a height of about eight feet. We also placed a yardstick on the background wall so that we could calibrate the measurements we would make from the frames of the video. We made videos of the free fall flight of a stack of 1, 2, 3, and 4 coffee filters. We then imported the frames of these videos into TEMATH and measured the position of the coffee filter in each frame of its flight. The differential equation that models free fall flight with air drag is[4]

$$m\frac{dv}{dt} = mg - \frac{1}{2}D\rho Av^2$$

where m is the mass, ρ is the density of air, A is the projected cross-sectional area of the falling object, D is the drag coefficient, g is the acceleration due to gravity, v is the velocity, and t is time. After being unsuccessful in fitting this

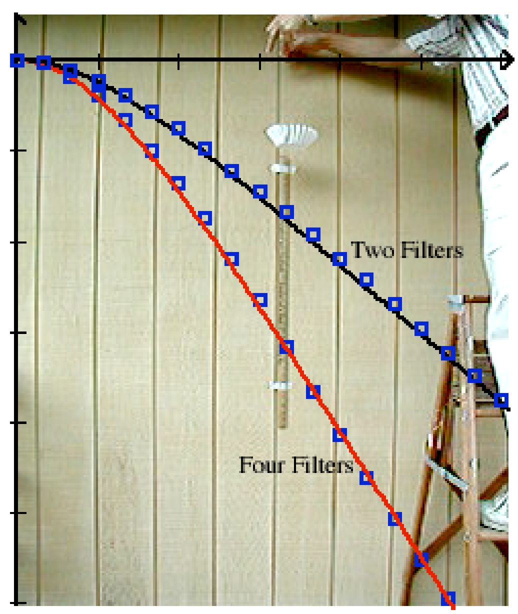

Figure 7 Modeling Free Fall Flight

model to the measured data, it was determined that the usually neglected upward buoyant force must be included in the model. The differential equation with the buoyant force is $m\frac{dv}{dt} = mg - \frac{1}{2}D\rho Av^2 - \rho gV$, where V is the volume of the object. Example models and measured data for the distance fallen by two and four filters are shown in Figure 7 above.

Free Fall Flight of a Ball

A standard model presented in most first-year mathematics courses is that of the free fall flight of a ball given by $s(t) = -\frac{1}{2}gt^2$. This model appears to be straightforward, however, our first attempt was not immediately successful. The digital camera we used for the videos takes 15 frames per second, that is, a picture is taken every 1/15 of a second or every 0.0667 milliseconds. Thus, the actual instant the ball is dropped could take place in between the frames of the video. When this happens, the data or the model must account for a time shift. As a result, we used the shifted model $s(t) = -\frac{1}{2}g(t - t_0)^2$, where s is the negative distance fallen, g is the acceleration due to gravity, t_0 is the time shift, and t is time. An example model and measured data is shown in Figure 8.

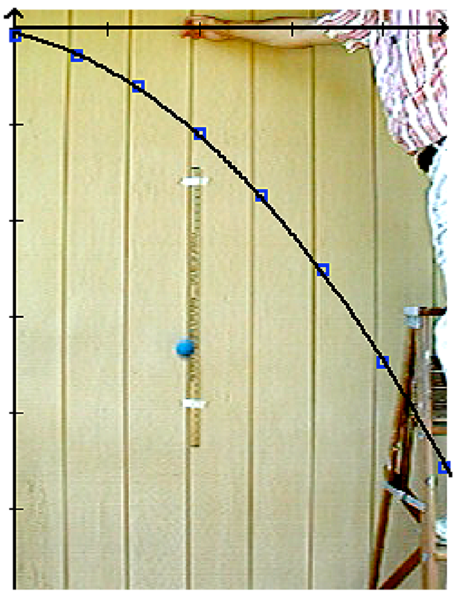

Figure 8 Free Fall Flight of a Ball

TEMATH (Tools for Exploring Mathematics)

TEMATH (Tools for Exploring Mathematics) is a mathematics exploration environment useful for investigating a broad range of mathematical problems. It is effective for solving problems in pre-calculus, calculus, differential equations, linear algebra, numerical analysis, and math modeling. TEMATH contains a powerful grapher, a matrix calculator, an expression calculator, a differential equation solver, a facility for handling and manipulating data, numerical mathematical tools, visual and dynamic exploration tools, and the capability for importing digital background images. TEMATH requires an Apple Macintosh computer running MAC OS 8.5-9.2.2 or Mac OS X (as a classic application), a 12" or larger monitor screen, 3 MB of free RAM, and 2MB of disk space (for TEMATH plus its support files). You can download a copy of TEMATH and its documentation, application files, and games from:

www2.umassd.edu/TEMATH

Bibliography

[1] GraphicConverter 3.9.1 (US) PPC, Lemke Software, Erich Heckel Ring 8A, 31228 Peine, Germany, www.lemkesoft.com.

[2] Robert Kowalczyk and Adam Hausknecht, *TEMATH - Tools for Exploring Mathematics Version 2.0.8*, 2002.

[3] Robert Kowalczyk and Adam Hausknecht, *Using TEMATH in Calculus*, 2000.

[4] R. A. Serway, Principles of Physics, Harcourt Brace College Publishing, 1994.