Name________________________________ **Student ID #** _________________________

Instructor____________________________ **Lab Period** _____ **Date Due** ___________

Lab 14
Sequences

Objectives

1. To graphically demonstrate the definition of the limit of a sequence.
2. To graphically investigate the properties of a sequence.

Limit of a Sequence

A sequence $\{a_n\}$ has the limit L if for every $\varepsilon > 0$ there is a corresponding integer N such that $|a_n - L| < \varepsilon$ whenever $n > N$.

Note that the value of N depends on the value of ε.

The goal of this laboratory is for you to develop a better understanding of sequences and their properties. To begin this lab, we need to set up *TEMATH* by doing the following:

- Select **Pen...** from the **Options** menu. Click the box containing the × to the left of **Highlight Selected Plot** to remove the ×. Click the **OK** button. All graphs will now be plotted with a thin line. This will make it easier to visualize the sequences.
- Select **Variables…** from the **Options** menu.
- Enter ***n*** for the independent variable name and enter ***a*** for the dependent variable name. Click **OK**. This changes *TEMATH*'s default variable names *x* and *y* to *n* and *a*, thus, making our explorations more intuitive since the common notation for an element of a sequence is a_n.

Exploration 1 *Visualizing the Definition of the Limit of a Sequence*

In this exploration, we want to examine the definition of the limit of a sequence from a graphical point of view. Let's use the example sequence $\left\{\frac{n}{n+1}\right\}$ for this exploration. To begin, let's generate the first 200 elements of this sequence by doing the following:

- Select **New Data Table** from the **Work** menu.

- Double-click the **Resize** command in the data table command list. Enter **200** into the command area and click the **enter** button or press the **Enter** key. This places an empty 200 row data table in the window.
- Select the ***n*** column. Double-click the **Fill** command. Enter **i** (where *i* is the row number) into the command area and press the **Enter** key. This fill function $n(i) = i$ places the integers 1, 2, 3, . . ., 200 into the *n* column.
- Select the ***n*** column. Double-click the **Generate** command. Enter the expression for the sequence **n/(n + 1)** into the command area and press the **Enter** key. This generating function will be applied to the values of *n* in the first column and it will place the corresponding values of the sequence into the ***a*** column. At this point, the ***a*** column contains the first two hundred terms of the sequence.
- To add (save) this sequence to the Work window, select **Add Data Table** from the **Work** menu. The table containing the sequence will appear as the first entry in the Work window.

To plot this sequence, do the following:

- Enter the domain $\mathbf{0 \le n \le 200}$ into the **Domain & Range** window.
- Select **Plot** from the **Graph** menu.

Examine this plot carefully. Does it appear that this sequence is approaching a limiting value? In fact, does it appear that this sequence is approaching 1 as *n* gets large? After you have examined the plot, go back to the very beginning of this lab and reread the definition of the limit of a sequence. Next, let's graphically apply this definition of the limit of a sequence to our example sequence to see if 1 could be the limit. To do this, follow these instructions:

- Select **New Constant** from the **Work** menu. Delete the default name of the constant and enter **e = 0.05**. The value of e will be the ε in the limit definition.
- Select **New Function** from the **Work** menu and enter **1** (the conjectured limit of the sequence).
- Select **New Function** from the **Work** menu and enter **1 + e**.
- Select **New Function** from the **Work** menu and enter **1 – e**.
- Select **Plot All** from the **Graph** menu.

Note that this places a band about the conjectured limit of the sequence and also note that any point of the sequence that enters this band is less than a distance of ε from the limit.

1. a) Does it appear that after some point in the sequence, all the elements of the sequence are contained within the ε band of the limit 1? ....

   b) What is the value of *N* that satisfies the limit definition for ε = 0.05? Move the cursor in the graph window to the point where the sequence enters the band and read the value of n from the "**n =**" cell located in the bottom portion of the Domain & Range window. If you need a closer look to find n, use *TEMATH*'s Zoom tool……..

The definition of the limit states that no matter how small we make ε, there should be some value of n after which all the elements of the sequence will enter and stay inside this band. Let's apply this definition by making ε smaller.

- Change the value of e to **e = 0.01** and select **Plot All** from the **Graph** menu.

2. a) Does it appear that after some point in the sequence, all the elements of the sequence are contained within this ε band of the limit 1?………………………………………

   b) What is the value of N that satisfies the limit definition for ε = 0.01?………………

Note that what we have done so far is not a proof that 1 is the limit of the sequence. It is only a visual representation of the definition of the limit which suggests that 1 may be the limit. For the limit to exist, the sequence must enter and stay within the ε band for any arbitrarily small value of ε and $n > N$, where N depends on the value of ε.

Exploration 2 *Gaining Experience with Sequences*

In this exploration, our goal is to examine many different types of sequences and to use them to help us develop a good intuition about sequences and their limits. For the sequences given below, we will use their plot to conjecture whether the sequence has a limit or not. Note that this is only a visual exercise that will enable us to guess the limit of a sequence and it is not a formal proof of the existence of the limit.

Example 1

Generate and plot the sequence $\left\{ \frac{1}{n} \right\}$ by doing the following:

- Click in the **Data Table** window to make it active (or select **Data Table** from the **Windows** menu).
- Select the ***n*** column and double-click the **Generate** command. Enter the generating function **1/n** into the command area and press the **Enter** key.
- Select **Add Data Table** from the **Work** menu.
- Select **Plot** from the **Graph** menu.

1. a) Does this sequence appear to have a limit?…….. If yes, what is the limit? ………..

   If no, then why not? ……………………………………………………………………

b) Is the sequence increasing, decreasing, or neither? ……………………………………

c) Does it appear that the sequence is bounded?…………………………………………

If so, what are the bounds? ……………………………………………………………

Example 2

- Generate and plot the sequence $\left\{ \frac{(-1)^n}{n} \right\}$.

2. a) Does this sequence appear to have a limit?…….. If yes, what is the limit? ………..

If no, then why not? ……………………………………………………………………

b) Is the sequence increasing, decreasing, or neither? ……………………………………

c) Does it appear that the sequence is bounded?…………………………………………

If so, what are the bounds? ……………………………………………………………

Example 3

- Generate and plot the sequence $\left\{ \cos\left(\frac{n\pi}{6} \right) \right\}$. Note: Press **Option p** for π.

3. a) Does this sequence appear to have a limit?…….. If yes, what is the limit? ………..

If no, then why not? ……………………………………………………………………

b) Is the sequence increasing, decreasing, or neither? ……………………………………

c) Does it appear that the sequence is bounded?…………………………………………

If so, what are the bounds? ……………………………………………………………

Example 4

- Generate and plot the sequence $\left\{ \frac{\cos(n)}{n} \right\}$.

4. a) Does this sequence appear to have a limit?…….. If so, what is the limit?…………

b) Is the sequence increasing, decreasing, or neither? ……………………………………

c) Does it appear that the sequence is bounded?……………………………………………

If so, what are the bounds? ……………………………………………………………

Example 5

- Generate and plot the sequence $\left\{ \dfrac{n}{\ln(n+1)} \right\}$.

5. a) Does this sequence appear to have a limit?…….. If so, what is the limit?…………

b) Is the sequence increasing, decreasing, or neither? ……………………………………

c) Does it appear that the sequence is bounded?……………………………………………

If so, what are the bounds? ……………………………………………………………

Example 6

- Generate and plot the sequence $\left\{ \dfrac{4n^3+3}{n^3+2n} \right\}$. Note: Be sure to use parentheses to enclose the numerator and the denominator when entering this expression into *TEMATH*.

6. a) Does this sequence appear to have a limit?…….. If so, what is the limit?…………

b) Is the sequence increasing, decreasing, or neither? ……………………………………

c) Does it appear that the sequence is bounded?……………………………………………

If so, what are the bounds? ……………………………………………………………

Example 7

- Generate and plot the sequence $\left\{ \dfrac{(-1)^n(3n^2-n-2)}{n^2+4n+1} \right\}$. Note: Be sure to use parentheses to enclose the denominator when entering this expression into *TEMATH*.

7. a) Does this sequence appear to have a limit?……. If so, what is the limit?…………

b) Is the sequence increasing, decreasing, or neither? ……………………………………

c) Does it appear that the sequence is bounded?…………………………………………

If so, what are the bounds? ……………………………………………………………

Exploration 3 *Sequences and Computer Arithmetic*

Sometimes computer arithmetic makes it more difficult to analyze a problem. For example, let's analyze the sequence $\{\sin(n\pi)\}$ by doing the following:

- Generate and plot the sequence $\{\sin(n\pi)\}$.

4. a) What is the range of this plot? (range = maximum a_n – minimum a_n) ……………

b) What does $\sin(n\pi)$ equal for any positive integer *n*?…………………………………

c) Does it appear that the computer is having trouble computing $\sin(n\pi)$ to machine accuracy? Explain your answer..……………………………………………………

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