Name________________________________ **Student ID #** _________________________

Instructor____________________________ **Lab Period** _____ **Date Due** ___________

Lab 15
Taylor Polynomials

Objectives

1. To develop an understanding for error bound, error term, and interval of convergence.
2. To visualize the convergence of the Taylor polynomials $P_n(x)$ to $f(x)$ as n increases.

Taylor's Formula

Let f be a function such that f and its first n derivatives are continuous on the closed interval I and let $f^{(n+1)}(x)$ exist on the interior of I. Then for c and x in I, there is a number ξ strictly between c and x such that

$$f(x) = P_n(x) + R_n(x)$$

where $P_n(x)$ is the n^{th} degree Taylor Polynomial given by

$$P_n(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

and $R_n(x)$ is Lagrange's form of the remainder term

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

An error bound for the approximation error E is given by

$$|E| = |f(x) - P_n(x)| \le \frac{M}{(n+1)!}|x-c|^{n+1}$$

where

$$|f^{(n+1)}(x)| \le M \text{ on the interval containing } c \text{ and } x.$$

Exploration 1 *Taylor Polynomials and the Sine Function*

In this exploration, we will use the sine function and its Taylor polynomials to help us develop a better understanding of the approximating properties of Taylor polynomials.

1. What is the Taylor Series for $f(x) = \sin(x)$ about $c = 0$?………………………………

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2. Find the following Taylor polynomials for $f(x) = \sin(x)$ about $c = 0$:

$P_1(x) =$ ……………………………………………………………………………

$P_3(x) =$ ……………………………………………………………………………

$P_5(x) =$ ……………………………………………………………………………

$P_7(x) =$ ……………………………………………………………………………

$P_9(x) =$ ……………………………………………………………………………

To begin this exploration, we need to set up *TEMATH* by doing the following:

- Click *off* autoscaling, set the domain to $-\pi \le \boldsymbol{x} \le \pi$ and set the range to $\mathbf{-2} \le \boldsymbol{y} \le \mathbf{2}$. Note: Press **Option p** for π.
- Select **Pen...** from the **Options** menu. Click the box containing the × to the left of **Highlight Selected Plot** to remove the ×. Click the **OK** button. All graphs will now be plotted with a thin line. This will make it visually easier to compare the approximating polynomials.
- Enter and plot $f(x) = \sin(x)$.
- Enter and overlay the plot of the first degree Taylor polynomial $P_1(x) = x$
- Enter and overlay the plot of the third degree Taylor polynomial $P_3(x) = x - \dfrac{x^3}{3!}$

Continue to enter and overlay the plots of higher degree Taylor polynomials until you find one that "visually" matches the graph of $f(x) = \sin(x)$ on the interval $-\pi \le x \le \pi$.

3. As n increases (n = 1, 3, 5, . . .), describe how the Taylor polynomials $P_n(x)$ become better approximations for $\sin(x)$. ……………………………………………………

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4. What is the lowest degree Taylor polynomial that "visually" matches the graph of $f(x) = \sin(x)$ on the interval $-\pi \le x \le \pi$?…………………………………………………

   Is this Taylor polynomial and $\sin(x)$ exactly the same? If not, by how much do they differ? To find out,

   - Click *on* autoscaling in the **Domain & Range** window.
   - Enter and plot the absolute value of the error function $|E_n(x)| = |\sin(x) - P_n(x)|$ for the Taylor polynomial given in question 4 above. Note: If, for example, $P_n(x)$ is $y6(x)$ in the Work window, you can enter $|E_n(x)|$ as **abs(sin(x) – y6(x))**.

5. a) What is the largest magnitude of the approximation error $|E_n(x)|$ on the interval $-\pi \le x \le \pi$? ……………………………………………………………………………

   b) Find an upper bound for the Lagrange remainder term $|R_n(x)|$. Show all your work and give reasons for each step……………………………………………………………

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   c) How does the error bound in part b) compare with the actual largest error (in magnitude) found in part a)? ……………………………………………………………

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6. a) What is the lowest degree Taylor polynomial that approximates the sine function on $-\pi \le x \le \pi$ correct to four decimal places, that is, you want $|E_n(x)| \le 0.5 \times 10^{-4}$?

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   b) Describe how you found this polynomial.……………………………………………

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You can use *TEMATH* to evaluate functions given in the Work window. For example, suppose you want to find the value of the function $y8(x)$ at $x = 3$, that is, you want to find $y8(3)$. To do this,

- Select **Calculators — Expression Calculator...** from the **Work** menu.
- When the Expression Calculator opens, enter the expression **y8(3)** and press the **Enter** key or click the **Evaluate** button. Be sure that the flashing cursor is on the same line as y8(3) when you press the Enter key. The value of y8(3) will be written to the next line.
- To find y8(17), simply edit the expression **y8(3)** to read **y8(17)** and press the **Enter** key.

You can also enter and evaluate expressions like $\sin(3) - y8(3)$ by doing the following:

- If you are not at the beginning of a blank line in the Expression Calculator, press the **Return** key.
- Enter the expression **sin(3) – y8(3)** and press the **Enter** key. The value will be written to the next line.

7. a) Using the polynomial you found in question 6, what is an approximate value for sin(2)? ……………………………………………………………………………………

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b) What is the error of this approximation? ………………………………………………

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Exploration 2 *Interval of Convergence*

The Taylor series for ln(*x*) about $x = 1$ is

$$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots + (-1)^{(n+1)}\frac{(x-1)^n}{n} + \cdots$$

$$= \sum_{i=1}^{\infty} (-1)^{(i+1)}\frac{(x-1)^i}{i}$$

In this exploration, we want to use graphs to visually estimate the interval of convergence of this Taylor series. To do this, we need to do the following:

- If autoscaling is *on*, click it *off*.
- Set the domain to $\mathbf{-1 \le x \le 3}$ and set the range to $\mathbf{-5 \le y \le 3}$.
- Enter and plot the function $f(x) = \ln(x)$.
- Enter the *TEMATH* expression **∑(i, 1, 10, (–1)^(i+1)(x–1)^i/i)** (press **Option w** for ∑) for the 10^{th} degree Taylor polynomial for ln(*x*) about $x = 1$.
- Select **Overlay** plot from the **Graph** menu.
- Enter the *TEMATH* expression **∑(i, 1, 25, (–1)^(i+1)(x–1)^i/i)** for the 25^{th} degree Taylor polynomial.
- Select **Overlay** plot from the **Graph** menu.
- Enter and overlay the plots of the 50^{th}, 75^{th}, and 100^{th} degree Taylor polynomials.

<u>Attach to this lab report a printed copy of this final graph</u>.

1. Using these graphs, visually estimate the interval of convergence of the Taylor series for ln(*x*) about $x = 1$. ……………………………………………………………………

   Give reasons why you selected this interval. ……………………………………………

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2. Using analytical tools, find the exact interval of convergence of the Taylor series for ln(*x*) about $x = 1$. Show all your work. …………………………………………………

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3. Find the Taylor series for $f(x) = \dfrac{1}{x+2}$ about $x = 0$ ……………………………………

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4. Using the graphical technique described above, visually estimate the interval of convergence of the Taylor series for $f(x) = \dfrac{1}{x+2}$ about $x = 0$ …………………………

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<u>Attach to this lab report a printed copy of the final graph used to obtain the answer for question 4</u>.

5. Using one of the standard tests for convergence, find the exact interval of convergence of the Taylor series for $f(x) = \dfrac{1}{x+2}$ about $x = 0$. Show all your work…………………

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6. In your own words, give a definition for "Interval of Convergence".……………………

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