LIGHTS, CAMERA, CALCULUS

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A digital camera makes it possible to record and import images of natural and man-made systems and objects into a computer. Once there, mathematical software enables students to analyze and model features of the image and, thus, the underlying structure and physics of the phenomena captured by the image. Digital technology can be easily integrated into the classroom in a manner that allows students to focus their energy on doing mathematics. The following explorations describe how this technology can be used in a calculus classroom to motivate students' learning of mathematics.

Finding the Volume of a Bowl

This exploration can be used when studying "Volumes of Revolution". Find a symmetrically shaped object and photograph it. In this example, we photographed a stainless steel mixing bowl. Download the image to your computer and paste the image on top of a pair of coordinate axes. We included a ruler in the picture and scaled the axes to match the inch increments on the ruler. In this way, the volume of the bowl can be computed in cubic inches. Ask your students to find a mathematical function that models the shape of the bowl and use it to estimate the volume of the bowl. The curve fitting the bowl shown in Figure 1 is the piecewise function

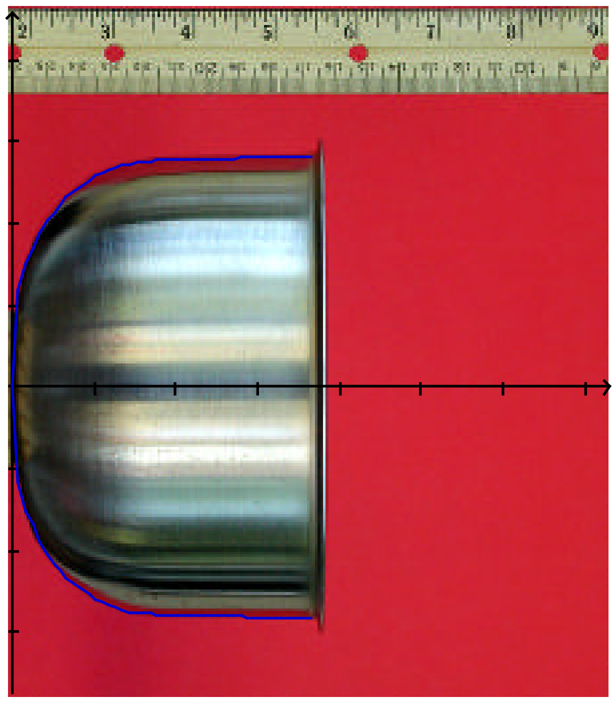


Figure 1 Finding the Volume of a Bowl

$$f(x) = \begin{cases} 2.7\sqrt[3]{x} & \text{if} \quad x \le 0.4 \\ 0.19x^3 - 1.1x^2 + 2.2x + 1.3 & \text{if} \quad 0.4 < x < 1.93 \\ 0.02x + 2.8 & \text{if} \quad 1.93 \le x \le 3.75 \end{cases}$$

Students can use the disc method to calculate the volume of the bowl by computing the integral $V = 2\pi \int_0^{3.75} [f(x)]^2 dx$ = 80.408 cubic inches = 1317.6 cc (ml). In this example, the thickness of the bowl is negligible and can be ignored. To build confidence in students with the disk method, you can have them check the accuracy of their volume calculation by filling the bowl with water and then pouring the water into a calibrated beaker (borrowed from the Chemistry Department). Before students can check their answer, they must convert their measurements to a common unit of measurement, in this case, we used cubic centimeters (cc) or equivalently milliliters (ml). The measured volume of the bowl was 1321 cc (ml) which is very close to the computed volume.

Modeling the Trajectory of Water from a Hose

For this exploration, you need to take a picture of a stream of water shooting out of a garden hose pointing at an upward angle. Care must be taken so that the pointing direction of the camera is perpendicular to the plane of the arc of water. You should also include a meter (yard) stick in the picture to serve as a reference of scale. After the picture is imported into the computer, you can paste the image on top of a pair of coordinate axes.

The objective of this exploration is to have your students use the equations of motion and parametric equations to trace the trajectory of the water. In this example, distances are measured in feet, velocity is measured in feet per second, and the standard (no external forces except gravity) parametric equations of motion are

$$x(t) = x_0 + v_0 \cos(\theta)t,\ y(t) = y_0 + v_0 \sin(\theta)t - (1/2)gt^2.$$

Students need to determine the values of the parameters x_0, y_0, v_0, and θ that will draw the pictured trajectory for the water. The values of x_0, y_0, and θ can be easily approximated from the picture by using TEMATH's Point tool and Line tool. Students can estimate v_0 by trial and error. Using the picture shown in Figure 2, the horizontal distances of the streams of water predicted by the standard equations of motion were always greater than the observed distances. This phenomenon leads to a discussion with the students as to why this is so. Hopefully, some students will suggest that when the picture was taken, a wind was blowing in the negative *x* direction providing an external force in that direction causing the horizontal distance of the water to be shortened. Using Newton's second law of motion, it can be shown that a term proportional to t^2 needs to be added to the $x(t)$ equation of motion, that is, $x(t) = x_0 + v_0 \cos(\theta)t - (1/2)a_w t^2$. Again, students can estimate the wind acceleration parameter a_w by trial and error. The parametric equations that were used for the fit shown in Figure 2 are

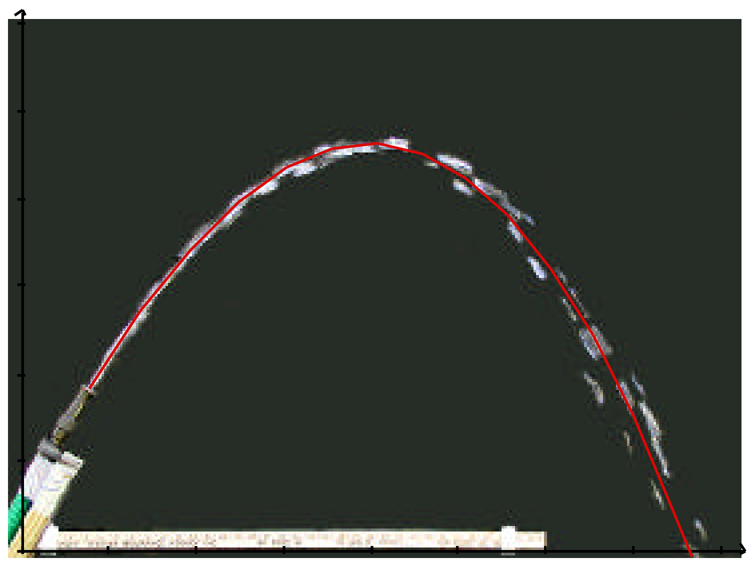


Figure 2 Trajectory of Water from a Hose

$$x(t) = 0.4 + 11\cos(1.03)t - 0.9t^2,\ y(t) = 0.92 + 11\sin(1.03)t - 16t^2.$$

Modeling Magnetic Field Patterns of Bar Magnets

Here is an experiment for a first course in differential equations. For this experiment, a collection of disk and bar magnets (or a strong horseshoe magnet), iron filings and a sheet of stiff paper are needed. The collection of disk and bar magnets are arranged to form a strong horseshoe magnet and the sheet of paper is placed over the magnet's two poles. Iron filings are sprinkled over the paper and the resulting pattern is then photographed.

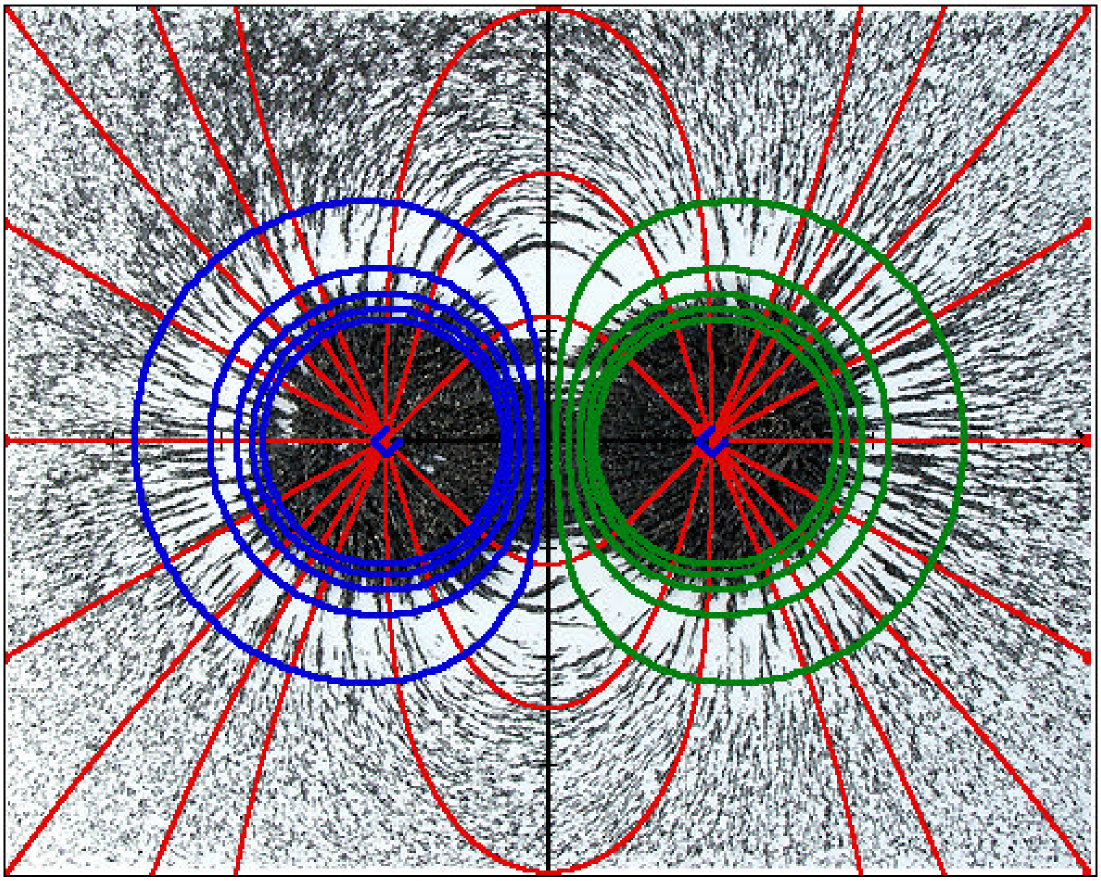


Figure 3 The Magnetic Field (curves through poles) of a Horseshoe Magnet

The objective is to compare the observed field patterns with those produced by inverse power law models for the field lines and find the model that best fits the observed phenomena. For example, the constant force curves of the horseshoe magnet (the circular curves around each pole in Figure 3) are assumed to be of the form

(E1) $$k = \|(x+c,y)\|^{-n} - \|(x-c,y)\|^{-n}$$

where $\|(a,b)\| = \sqrt{a^2+b^2}$ and the magnet's poles are located at $(\pm c, 0)$. Thus, we are assuming that a horseshoe magnet's field can be modeled by two "point monopoles". Using implicit differentiation on E1, we find

(E2) $$\frac{dy}{dx} = \frac{(x+c)\|(x-c,y)\|^{n+2} - (x-c)\|(x+c,y)\|^{n+2}}{y\left(\|(x+c,y)\|^{n+2} - \|(x-c,y)\|^{n+2}\right)} = F(x,y)$$

Since the magnet field lines (curves passing through the poles in Figure 3) are orthogonal to the constant force curves, they can be found by using TEMATH's differential equation solver to plot the solutions of $dy/dx = -1/F(x,y)$. Based on a model of a magnet field of a coil in Section 30-5 of Halliday-Resnick-Walker (see [3]) and the discussion of magnetic fields of a localized current distribution in Section 5.6 of J. D. Jackson [see 4], we decided to try $n = 3$. The close-up view (Figure 4) shows that the nearly straight magnet field lines are orthogonal to the circular constant force curves and are approximately parallel to the iron filings. In Figure 5, a magnet field line through a fixed point is plotted for each of the models corresponding to $n = 1, 2, 3, 4$. Figures 3-5 together provide qualitative evidence that the model for $n = 3$ is a plausible approximation to a true model. However, because of the relatively large size of the iron filings, the friction between the filings and the paper, and the relatively crude method of applying the filings by sprinkling them onto the paper by hand, it would be unwise to conclude more!

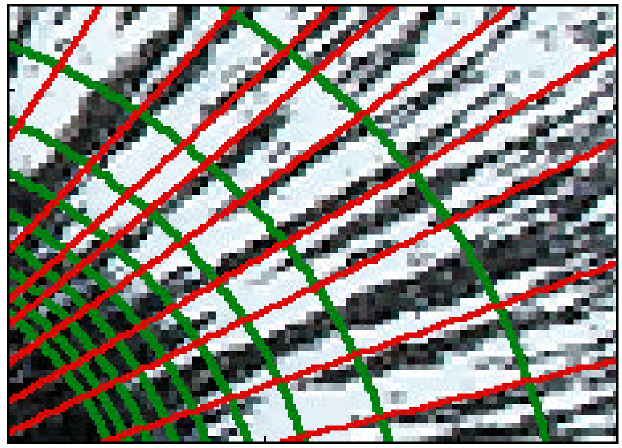


Figure 4 Zoomed View of the Magnet's Field

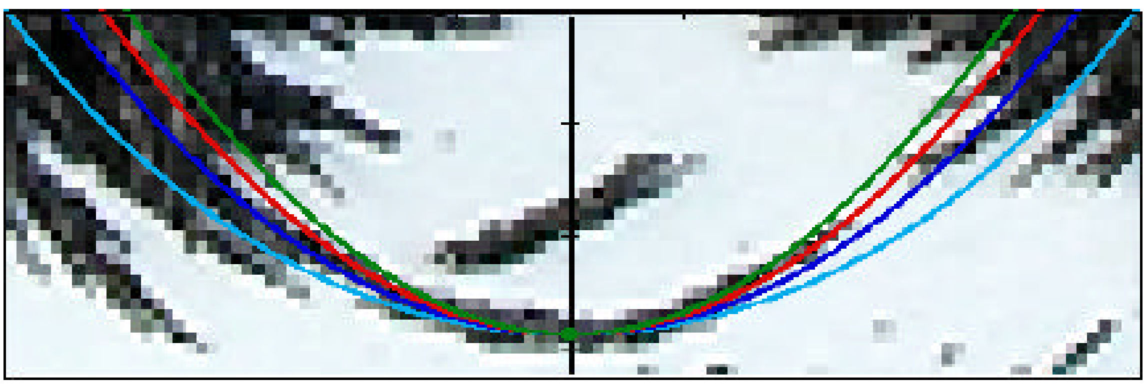


Figure 5 Magnet Field Lines for n = 1, 2, 3, 4 (ordered from bottom to top)

Modeling the Thread of a Wood Screw

We used a 50x digital microscope connected to a computer to photograph a wood screw (see Figure 6). Though difficult, care was taken to hold the screw so that its long axis was both horizontal and parallel to the microscope's view plane. Next, the x,y-projection

$$(x,y) = (t, R\sin[\frac{2\pi}{P}(t+w)])$$

of the helix

$$(x,y,z) = (t, R\sin[\frac{2\pi}{P}(t+w)], R\cos[\frac{2\pi}{P}(t+w)])$$

is used to model the screw's thread. The image was plotted over $0 \le x \le 2$, $-1 \le y \le 1$. Thus, the units in both directions are the same. Using TEMATH's Line Tool, it is easy to determine a value of R (the radius of the screw) and a value of P (the distance between successive turns of the thread), and w (the horizontal shift). In this case, we found that $R = 0.89$, $P = 0.75$, and $w = 0.135$.

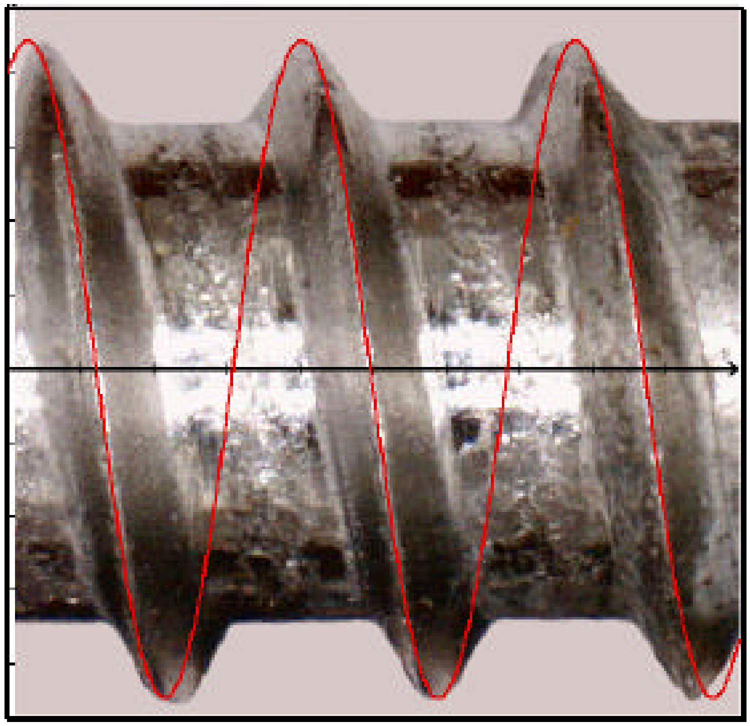


Figure 6 Model of a Wood Screw Thread

TEMATH (Tools for Exploring Mathematics)

TEMATH (Tools for Exploring Mathematics) is a mathematics exploration environment useful for investigating a broad range of mathematical problems. It is effective for solving problems in pre-calculus, calculus, differential equations, linear algebra, numerical analysis, and math modeling. TEMATH contains a powerful grapher, a matrix calculator, an expression calculator, a differential equation solver, a facility for handling and manipulating data, numerical mathematical tools, and visual and dynamic exploration tools. TEMATH requires an Apple Macintosh computer (post MacPlus; PowerPC even better) running MAC OS 7.5-9.1, a 12" or larger monitor screen, 3 MB of free RAM, and 2MB of disk space (for TEMATH plus its support files). You can download a copy of TEMATH and its documentation, application files, and games from:

www2.umassd.edu/TEMATH

Bibliography

[1] Robert Kowalczyk and Adam Hausknecht, *TEMATH - Tools for Exploring Mathematics Version 2.0.6*, 2000.

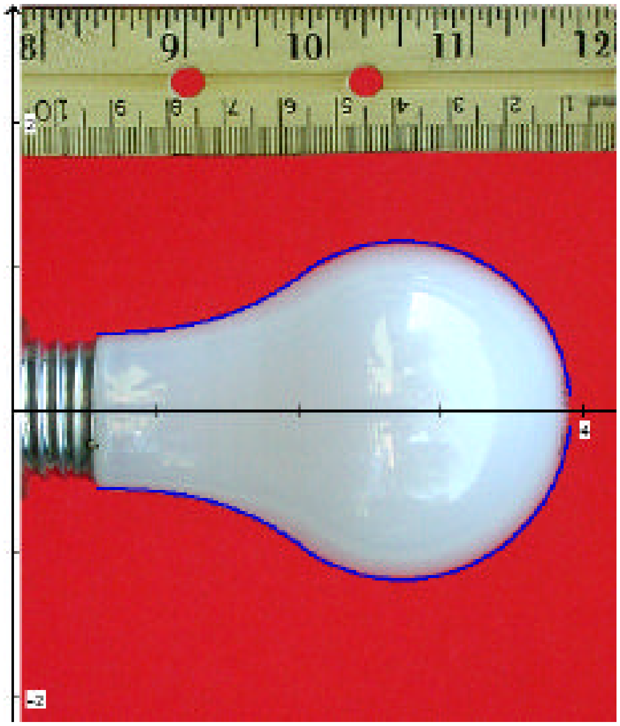
[2] Robert Kowalczyk and Adam Hausknecht, *Using TEMATH in Calculus*, 2000.

[3] Halliday, Resnick, and Walker, *Fundamentals of Physics*, Fifth Edition, Wiley, 1997.

[4] J. D. Jackson, *Classical Electrodynamics*, Second Edition, Wiley, 1975.

More Explorations

Estimating the Volume of Glass Needed to Make a Light Bulb



This exploration can be used when you're presenting the theory for the "area of a surface of revolution" in your calculus class. Motivate your students by asking them, "How much glass is needed to make a light bulb?" Once you have your students thinking about the problem, use a digital camera to take a picture of a light bulb taking care not to distort the perspective — you need an accurate image of the projected 2D shape of the bulb. Include a ruler in the picture so that accurate measurements can be made from the picture. After the image is downloaded into the computer, ask your students to find a mathematical expression for a curve that fits the shape of the upper half of the bulb — this will usually be a piecewise function. Students can check the accuracy of the curve by plotting it on top of the image. It will probably take several tries to find a suitable mathematical function that fits the shape of the bulb. Using this function, students can find the surface area of the glass part of the bulb. In order to estimate the volume of glass needed to make the bulb, you need to measure the thickness of the glass. To do this, you must carefully break a bulb, borrow a micrometer (from the Physics Department) and use it to measure the thickness of a piece of glass from the broken bulb. Using the calculated surface area of the bulb and the measured thickness of the glass, students can estimate the volume of glass in the light bulb. In a particular example that we tried, the function fit to the shape of the bulb was

$$f(x) = \begin{cases} 0.1x^3 - 0.135x^2 + 0.075x + 0.521 & \text{if} \quad 0.58 \le x \le 2 \\ \sqrt{1.18^2 - (x - 2.72)^2} & \text{if} \quad 2 < x < 3.9 \end{cases}$$

To build students' confidence in the model, you can have them compute the arc length around the top and bottom of the bulb and compare it to the "real" arc length of the bulb measured with a flexible cloth tape measure. In our case, the computed arc length was 8.23 inches and the measured arc length was 8.25 inches — a good confidence builder indeed! Using $f(x)$ to compute the surface area, we obtained

$$S(x) = 2\pi \int_{0.58}^{3.9} f(x)\sqrt{1 + [f'(x)]^2}\,dx = 20.3 \text{ square inches}$$

The measured thickness of the glass of the bulb was 0.024 inch and the approximate volume of glass needed to make the bulb was $20.3 \times 0.024 = 0.4872$ cubic inches.

Modeling the Exponential Growth of a Tibia Curta Shell

Nature provides many wonderful opportunities to use mathematics to model the physical characteristics of a living organism. The goal of this exploration is to model the growth of a Tibia Curta shell. TEMATH's Line tool was used to measure the lengths of the cylindrical growth segments of the shell. To give a frame of reference for the measurements, we first drew a line across the center of the entire shell and then measured the lengths of the spirals along this line (see Figure 8). The measurements were entered into a data table and plotted. Noting that the plotted data looked exponential, TEMATH's Least Squares Exponential Fit tool was used to find the exponential fit

$$f(x) = 0.113536951917\, e^{1.93526432846\, x}$$

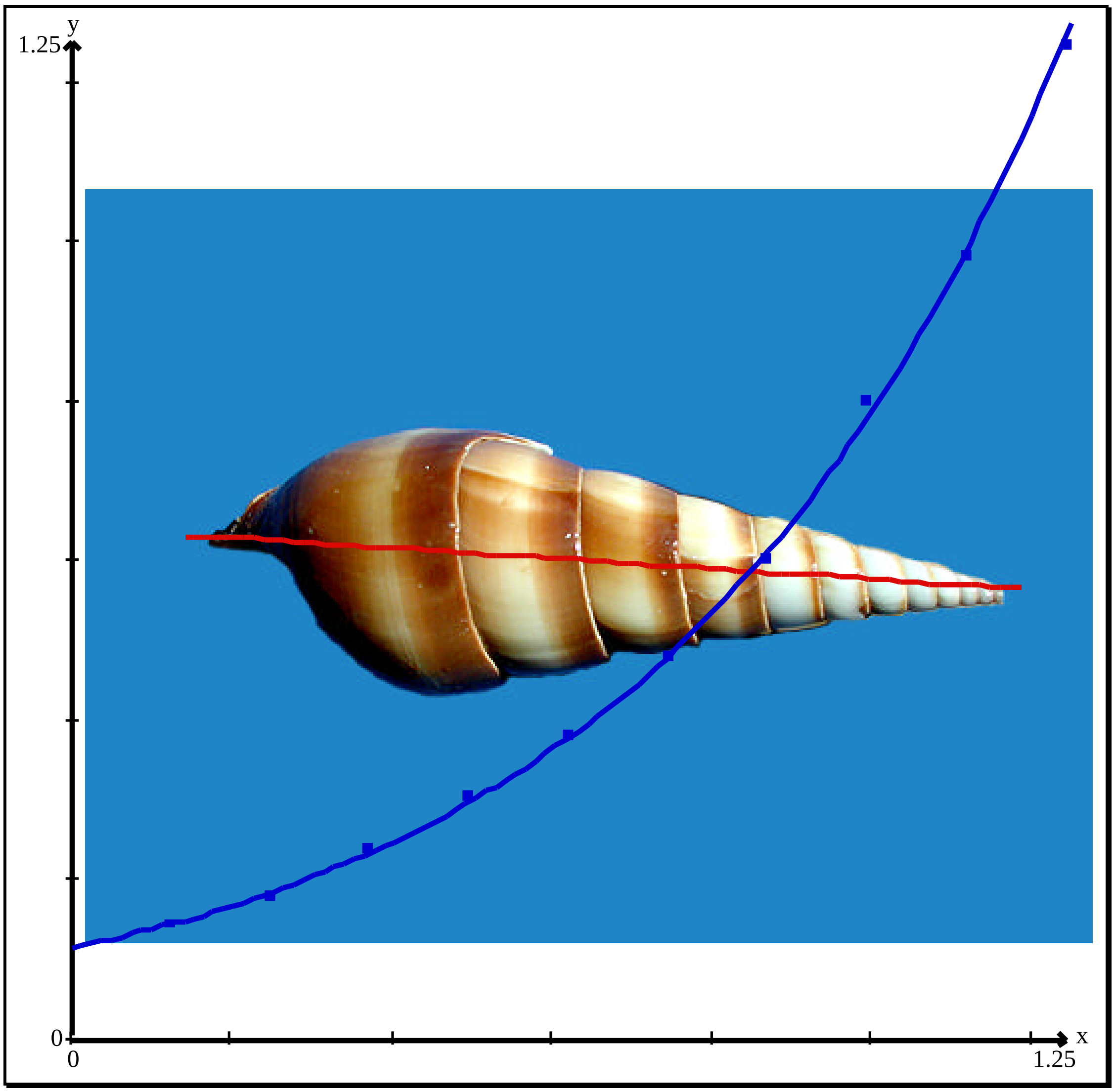


Figure 8 Exponential Growth of a Shell

Modeling the Spiral of a Chambered Nautilus Shell

Polar coordinates can be used to model the spiral structures found in many shells, in particular, the Chambered Nautilus. We used a digital camera to take a picture of a Chambered Nautilus shell, imported the image into a computer, and copied it into TEMATH's Polar Plot Mode. Using TEMATH's Point tool, we sampled points along the shell's spiral at intervals of $\pi/4$ radians (see Figure 9). Next, we created a table of values for these points, found the least squares exponential fit $r = 0.0693867881740e^{0.169139870975t}$ using

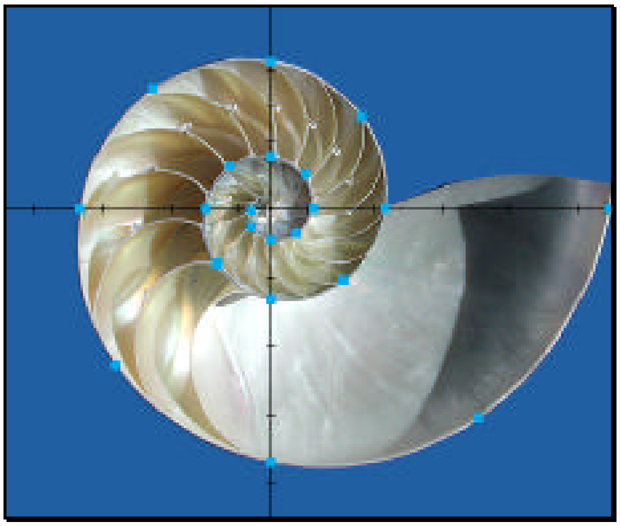


Figure 9 Nautilus Shell Data Sampling

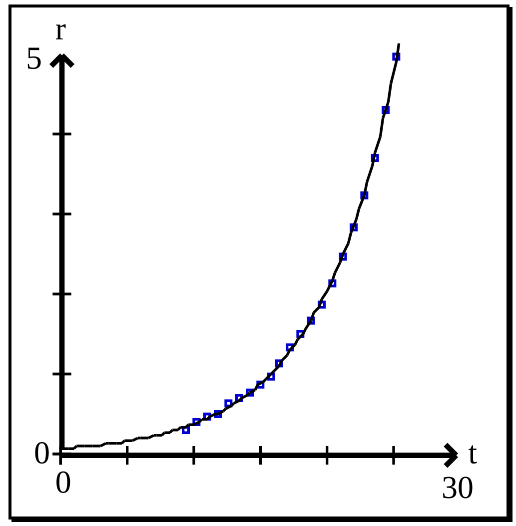


Figure 10 Least Squares Exponential Fit

rectangular coordinates (see Figure 10), and overlaid the polar plot of the fit on top of the image of the shell (see Figure 11). The fit is excellent! Real-life applications truly make our students appreciate the modeling potential of mathematical functions.

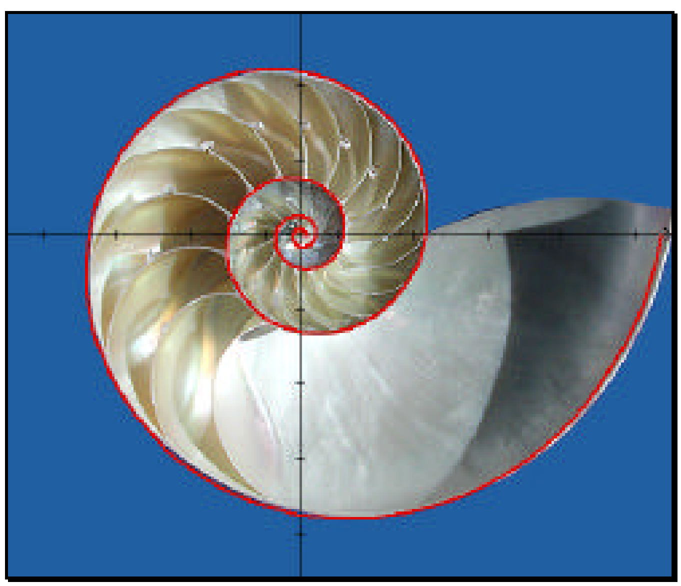


Figure 11 Nautilus Shell Spiral Fit

Modeling the Shape of Leaves with Polar Curves

Many leaves in nature are symmetric in shape and can be easily modeled by mathematical functions. In this example, the leaf reminded us of a cardiod so we tried polar equations to model the leaf. We quickly observed that the standard cardiod equations did not accurately model the leaf. We then sampled data points along the top edge of the leaf from $t = 0$ to $t = 2.25$ radians and fit the resulting data with an eighth degree polynomial using the technique of least squares. From $t = 2.25$ to $t = \pi$ radians, the leaf turns back in on itself, and, thus, can not be modeled by a single function. Since we wanted a single piecewise function to model the leaf, we decided to extend the polynomial to the pole ($2.25 \le t \le 2.36$) and set $r(t) = 0$ on the remaining part of the interval. The bottom half of the leaf was modeled by a reflection of the top half curve. The final piecewise curve fit we obtained is given by

$$r(t) = \begin{cases} p(t) & \text{if} \quad 0 \le t \le 2.36 \\ 0 & \text{if} \quad 2.36 < t \le 2\pi - 2.36 \\ p(2\pi - t) & \text{if} \quad 2\pi - 2.36 < t \le 2\pi \end{cases}$$

where

$$p(t) = -1.12t^8 + 10.04t^7 - 37.07t^6 + 72.29t^5 - 79.63t^4 + 48.79t^3 - 14.6t^2 - 0.180t + 4.46$$

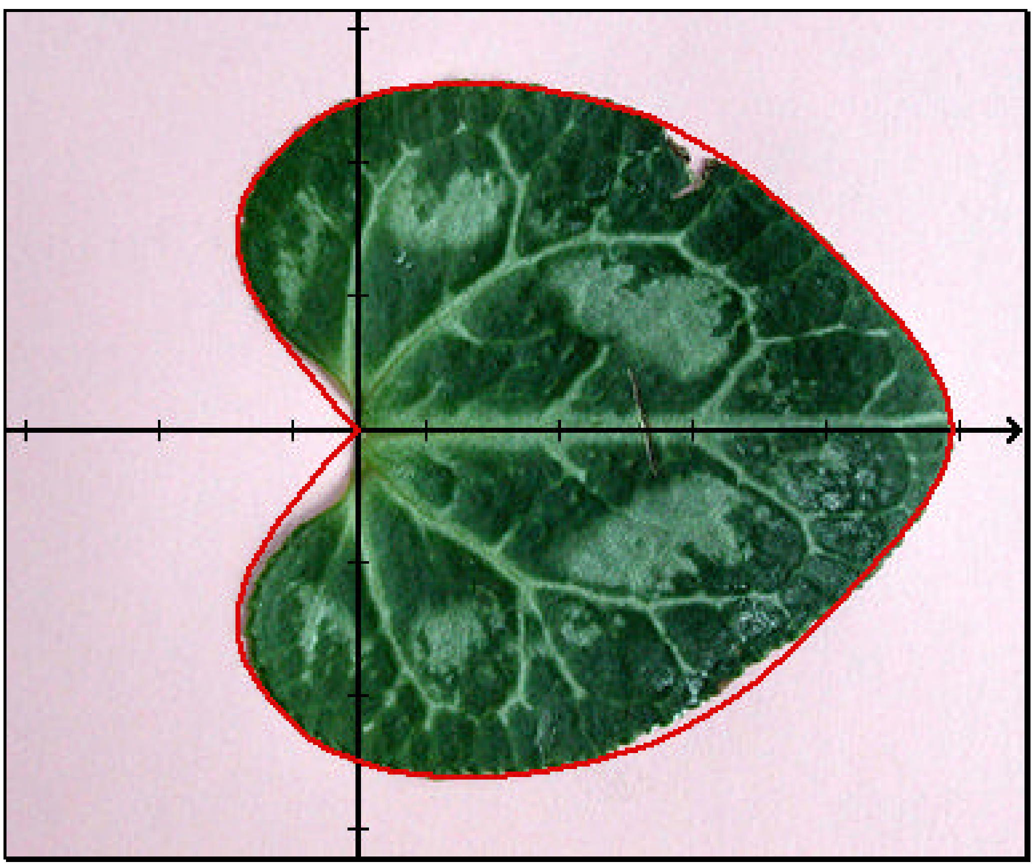


Figure 12 Modeling the Shape of Leaf with a Polar Curve

Using Vectors to Model a Starfish

This example provides a practice environment for students to use vectors to measure the angle between the "arms" of a starfish. TEMATH's Line tool is used to draw lines from the origin to the centers of the ends of the arms. The Point tool is then used to place a point at this center position and record the coordinates of the two points. This gives the position coordinates of the vectors. In this example, the two vectors are

$$\vec{v}_1 = \langle 3.08, 1.08 \rangle \quad \text{and } \vec{v}_2 = \langle 2.04, -2.66 \rangle$$

If θ is the angle between the two vectors, then

$$\cos(\theta) = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1||\vec{v}_2|} = 0.3117$$

So the angle between the two vectors (and the two arms of the starfish) is

$$\theta = \cos^{-1}(0.3117) = 1.2538 \quad (\text{or } 71.84^\circ)$$

For a perfectly symmetrical five-armed starfish, the angle between the arms would be 72˚.

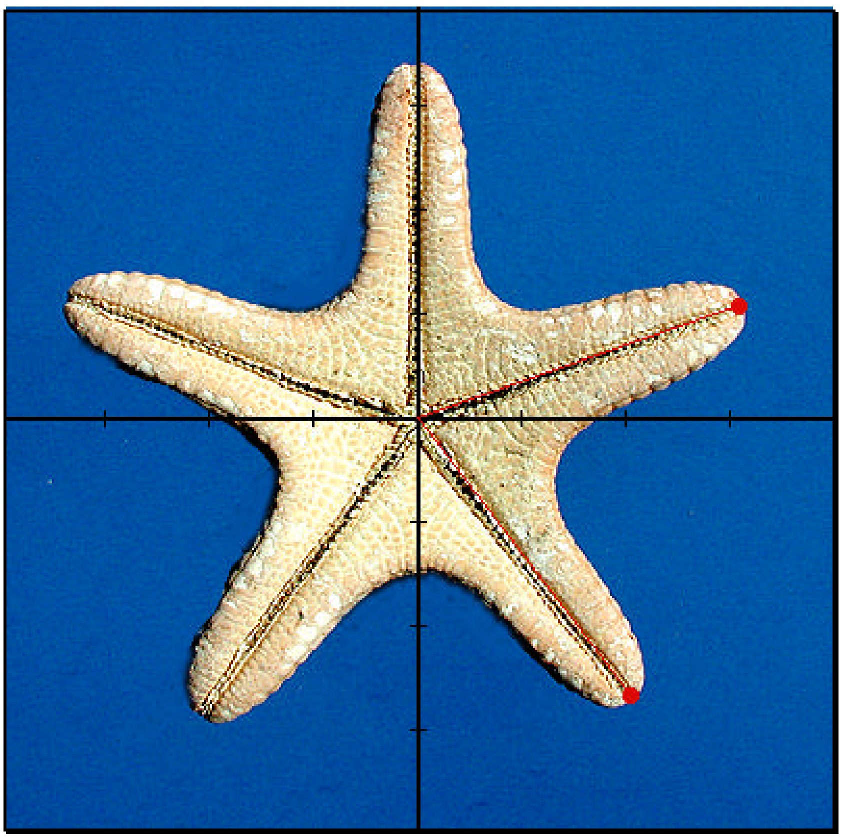


Figure 13 Using Vectors to Model a Starfish

Using Torricelli's Law to Model the Water Discharge from a Hole in a Bottle

For this experiment, we obtained a plastic bottle, drilled five holes into the bottle (3/16 in. in diameter), filled the bottle with water, and used our digital camera to record the outflow of the water through the holes. Rulers were placed both horizontally and vertically in the picture to ensure that the picture was not distorted and that the axes could be scaled properly. The goal of this experiment is to verify Torricelli's law that the velocity of the water leaving the hole of a bottle is given by $v = \sqrt{2gh}$, where h is the height of the water above the hole. Since $H - h = \frac{1}{2}gt^2$, the horizontal distance of the water flow from the hole is $x = vt = \sqrt{2gh}\sqrt{\frac{2(H-h)}{g}} = 2\sqrt{h(H-h)}$. We used TEMATH's Point tool to measure H, h for the five holes, and x for the five outflows of water. When we tried fitting Torricelli's model to the data measured from the picture shown in Figure 14, Torricelli's law always predicted greater horizontal distances.

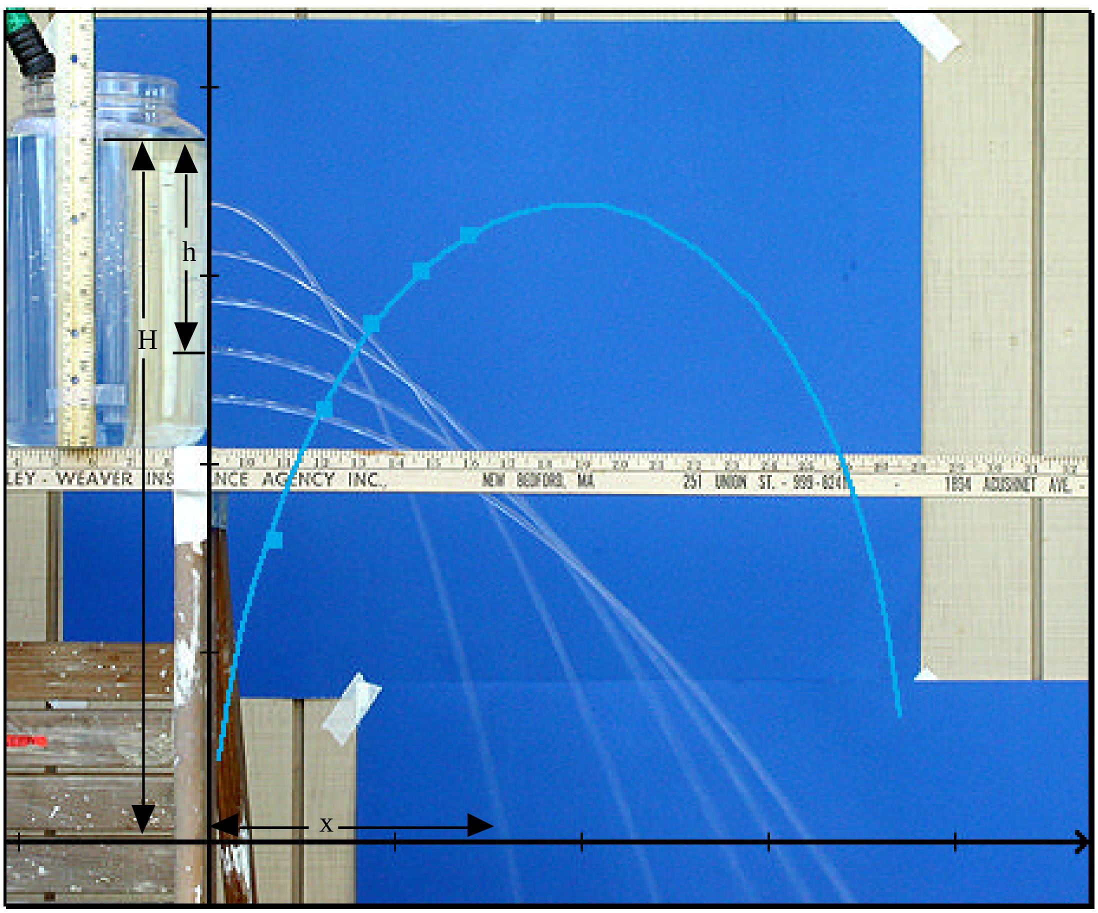


Figure 14 Using Torricelli's Law

We then realized that Torricelli's law is for nonviscous ideal fluids. Upon consultation with a Mechanical Engineering colleague, we adjusted the model for the jet contraction at the hole and the orifice discharge coefficient. These modifications to the velocity model give $v = c_c c_v \sqrt{2gh}$, where c_c is the contraction coefficient and c_v is the discharge coefficient. This model fit the data better, but it still wasn't a good fit. We were still missing something. Studying Bernoulli's equations led us to include another term in the model to compensate for the difference in pressures at the hole and the top of the bottle. Our final model was

$$x = 2c_c c_v \sqrt{(h - p)(H - h)}$$

where p is the constant that models the pressure effect. This modified model fits the data well (see Figure 14).

The measured data for the height h of the water above the hole and horizontal distance x traveled by the water are given in the following table:

h (inches)	**x (inches)**
1.7696	7.9896
3.0968	11.4656
4.3608	13.7408
5.6880	15.1312
6.9520	16.0792

If you have any suggestions for a better model, please contact us.